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THREE-DIMENSIONAL STRESS ANALYSIS OF A LAMINATED PLATE CONTAINING AN ELLIPTICAL CAVITY

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An analysis procedure is developed to study the stress distribution in a laminated plate containing an elliptical cavity under tenside loading and a uniform temperature change. Also considered is the stress field around a radial crack emanating from a hole in a laminate. The analysis follows from an extension of a previously developed finite element approach.

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PREFACE

In this work, a previously developed solution, based on a higher order finite element simulation, is extended to treat the problem of a composite laminate containing an elliptical cavity. Loading conditions consist of axial tension plus a uniform temperature change. This class of problems is fundamental to an understanding of local failure mechanisms which prevail in the vicinity of cutouts or cracks in composite laminates, where the influence of lamination geometry must be recognized. Furthermore, the inclusion of thermal stresses in the analysis allows the calculation of curing stresses induced in the fabrication process. Computer program listing and input data instructions for this stress analysis routine may be obtained from Dr. N. J. Pagano (AFML/MBM), the project monitor for this program.

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SECTION I
INTRODUCTION

One of the unique characteristics of laminated composite materials is the delamination mode of failure that occurs at free edges. This behavior has been investigated experimentally and by mathematical stress analysis models for laminates with straight free edges. Experimental studies by Pipes, Kaminski, and Pagano[1]* have shown that the laminate stacking sequence can affect the static strength of laminates. Similar effects of stacking sequence were found by Foye and Baker[2] for fatigue loading conditions. Mathematical stress analyses of straight free edges of laminate plates have shown that changes in stacking sequence can change the behavior of the interlaminar stresses. In particular, investigators[3,4] have found that the sign of the transverse normal stress σ_z changes from tension to compression as the stacking sequence is changed. Furthermore, Pagano and Pipes[6,7] have shown that high tensile stresses are associated with the decreased laminate strengths reported in References [1] and [2]. These observations point to the importance of understanding the stress behavior of laminates near a free edge. Through such an understanding, Lackman and Pagano[8] have successfully reduced interlaminar stresses at the free edge to alleviate the effects of the delamination mode of behavior.

While the straight free edge can be studied as a problem with two-dimensional variations in stresses and displacements, the curved free edge is inherently a three-dimensional problem. Thus, the circular hole problem is more difficult to treat in terms of a stress analysis than the straight free edge problem. Dana and Barker[9] have investigated the three-dimensional stress analysis of four-ply $(0/90)_s$ and $(45/-45)_s$ laminates with circular holes and found that the sign and magnitude of the σ_z distribution around the hole can change with changes in stacking sequence. Rybicki and Hopper[10] have developed an analysis for the interlaminar stress distributions for a six-ply symmetric laminate containing a circular hole. Experimental investigations by Daniel, Rowlands and Whiteside[11] on the effects of stacking sequence on the strength of laminated plates with holes indicated that variations in stacking sequence effected the laminate strength and failure mode. Other studies for example by Greszczuk[12], Waddoups, et al.[13], and Waszczuk and Cruse[14] have predicted failure of a composite plate containing a hole and made comparisons with experimental

data. These studies are a first step toward understanding the behavior of laminates containing a circular hole. The stress analyses for these studies did not include interlaminar stresses. Two basic three-dimensional problems that can provide insight into understanding laminate behavior near a curved free edge are considered here. One problem is the state of stress that exists around a circular hole. The other problem is the behavior around a noncircular hole such as an elliptical hole.

In the following, a description of an analysis procedure to treat stresses around a circular or elliptical hole in a laminated plate due to a uniform loading distant to the hole is described. The analysis procedures have been implemented into an existing three-dimensional finite element stress analysis [10] and numerical results for example cases are presented.

SECTION II
METHOD OF STRESS ANALYSIS

The method of stress analysis is based on the equilibrium finite element analysis described in Reference [10]. The details of the analysis are given in that reference. A summary of the analysis is given here as background information and extensions are described in detail.

In the analysis, each ply is modeled as a homogeneous orthotropic material with linear stress-strain behavior. Symmetric laminates are considered with applied stress loading conditions distant to the hole. The analysis is based on a three-dimensional equilibrium finite element representation for the stresses. Three Maxwell stress functions are associated with each element to satisfy equilibrium exactly within each element. Equilibrium between contiguous elements is satisfied exactly by equating unknown coefficients between elements. All stress boundary conditions are satisfied exactly. In the equilibrium approach, displacement boundary conditions and compatibility are satisfied approximately.

The following sections contain a description of the extensions of the existing analysis to treat an elliptical hole in a laminated plate. The results of interest for the elliptical hole problem are the interlaminar stresses.

In Reference [15], Rybicki and Hopper examined an approach to transform the problem of a plate containing an elliptical hole into a problem of a plate containing a circular hole. This is accomplished by starting with the conformal transformation for mapping an ellipse into a circle. The transformation is applied to the governing equations of the elliptical hole problem. The result is a description of an equivalent circular hole problem with different material properties and loading conditions. A transformation of the circular hole solution, back to the elliptical hole problem, is obtained based on the inverse of the original transformation. This approach was extended to the three-dimensional case. The procedure is given in the following.

A typical lamina is shown in Figure 1. The stress-strain relations for each lamina of the laminate have the form

$$
\begin{aligned}
\epsilon_x &= S_{11}\sigma_x + S_{12}\sigma_y + S_{13}\sigma_z && && + S_{16}\tau_{xy} + \alpha_1\Delta T\\
\epsilon_y &= S_{12}\sigma_x + S_{22}\sigma_y + S_{23}\sigma_z && && + S_{26}\tau_{xy} + \alpha_2\Delta T\\
\epsilon_z &= S_{13}\sigma_x + S_{23}\sigma_y + S_{33}\sigma_z && && + S_{36}\tau_{xy} + \alpha_3\Delta T\\
\gamma_{xz} &= && S_{44}\tau_{xz} + S_{45}\tau_{yz} && \\
\gamma_{yz} &= && S_{45}\tau_{xz} + S_{55}\tau_{yz} && \\
\gamma_{xy} &= S_{16}\sigma_x + S_{26}\sigma_y + S_{36}\sigma_z && && + S_{66}\tau_{xy} + \alpha_6\Delta T
\end{aligned}
\tag{1}
$$

where the S_{ij}'s are the compliance coefficients, σ's and τ's are the stresses, and $\alpha_i\Delta T$'s are the strains due to temperature change. The stresses are related to the Maxwell stress functions χ_1, χ_2, and χ_3 by

$$
\begin{aligned}
\sigma_x &= \frac{\partial^2 \chi_2}{\partial z^2} + \frac{\partial^2 \chi_3}{\partial y^2} & \tau_{xz} &= -\frac{\partial^2 \chi_2}{\partial x \partial z}\\
\sigma_y &= \frac{\partial^2 \chi_1}{\partial z^2} + \frac{\partial^2 \chi_3}{\partial x^2} & \tau_{yz} &= -\frac{\partial^2 \chi_1}{\partial y \partial z}\\
\sigma_z &= \frac{\partial^2 \chi_1}{\partial y^2} + \frac{\partial^2 \chi_2}{\partial x^2} & \tau_{xy} &= -\frac{\partial^2 \chi_3}{\partial x \partial y}
\end{aligned}
\tag{2}
$$

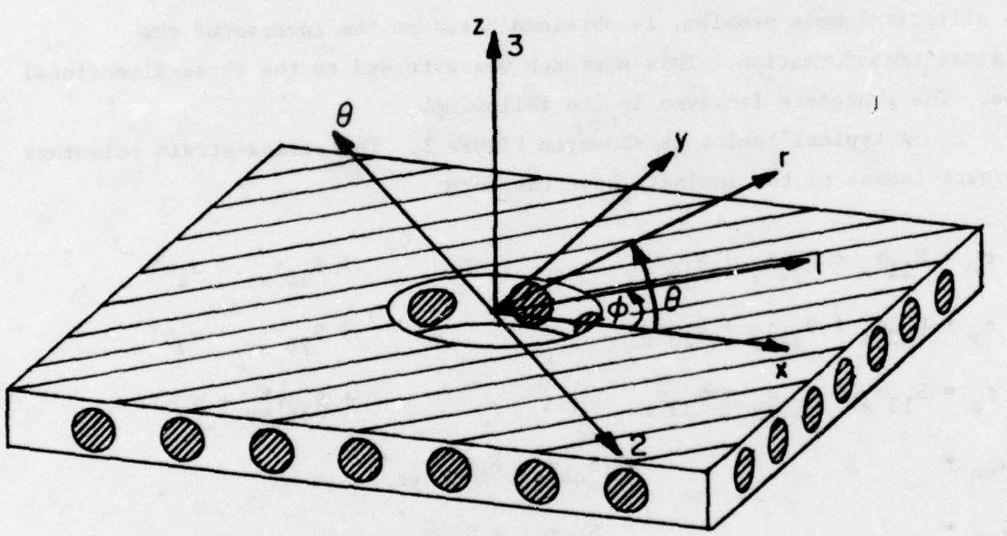

FIGURE 1. PRINCIPAL MATERIAL DIRECTIONS (1, 2, and 3), (x, y, and z), AND (r, θ, and z) COORDINATE SYSTEMS FOR A TYPICAL LAMINA

The first step is to express the strains of Equation (1) in terms of the stress functions, x_i, by combining Equations (2) and (1). Next, the compatibility equations are expressed in terms of the stress functions. The transformation

$$\begin{aligned} x &= x' \\ y &= \beta y' \\ z &= z' \end{aligned} \tag{3}$$

was applied to all the equations. One axis of the ellipse lies along the x-axis. This axis has a length of 1.0. The value of β is then the length of the axis of the ellipse in the y-direction. The (x',y',z') system pertains to the equivalent laminate with a circular hole and different material properties. For brevity, only the results of the transformation are listed. The material properties for the circular hole problem (S'_{ij} and α'_i) are related to those for the elliptical hole problem (S_{ij} and α_i) and to β as shown in the following equations.

$$\begin{aligned}
&& S'_{11} &= S_{11}/\beta^4 \\
&& S'_{22} &= S_{22} \\
&& S'_{12} &= S_{12}/\beta^2 \\
&& S'_{13} &= S_{13}/\beta^4 \\
&& S'_{16} &= S_{16}/\beta^3 \\
&& S'_{23} &= S_{23}/\beta^2 \\
&& S'_{26} &= S_{26}/\beta \\
\alpha'_1 &= \alpha_1/\beta^2 & S'_{33} &= S_{33}/\beta^4 \\
\alpha'_2 &= \alpha_2 & S'_{36} &= S_{36}/\beta^3 \\
\alpha'_3 &= \alpha_3/\beta^2 & S'_{44} &= S_{44}/\beta^4 \\
\alpha'_6 &= \alpha_6/\beta & S'_{45} &= S_{45}/\beta^3 \\
&& S'_{55} &= S_{55}/\beta^2 \\
&& S'_{66} &= S_{66}/\beta^2
\end{aligned} \tag{4}$$

The stress functions are transformed as follows.

$$x_1' = x_1$$
$$x_2' = \beta^2 x_2 \qquad (5)$$
$$x_3' = x_3$$

where the primes denote the circular hole problem.

The relationship between the stresses for the circular hole problem and the elliptical hole problem are

$$\sigma_x = \sigma_x'/\beta^2$$
$$\sigma_y = \sigma_y'$$
$$\sigma_z = \sigma_z'/\beta^2$$
$$\tau_{xz} = \tau_{xz}'/\beta^2 \qquad (6)$$
$$\tau_{yz} = \tau_{yz}'/\beta$$
$$\tau_{xy} = \tau_{xy}'/\beta$$

where the primed stresses refer to the circular hole problem and the unprimed stresses refer to the elliptical hole problem.

The stress boundary conditions around the elliptical hole were transformed to boundary conditions for the equivalent circular hole problem. The algebra for this step is given in Appendix A while Appendix B summarizes the constraint equations to satisfy the stress-free conditions at the circular hole.

The procedure for including these transformations into the computer program starts with the elliptical geometry, the material properties and the loading conditions. Through Equations (4) and (6) the program transforms the properties and stress boundary conditions to those of an equivalent circular hole problem. After the solution is obtained, the stresses for the equivalent circular hole problem are evaluated in the rectangular (x', y', z') coordinate system. These stresses are transformed to the stresses for the elliptical hole in the (x, y, z) coordinate system by using Equations (6). Additional transformations of the Mohr's circle type were included in the program to obtain the tangential and interlaminar stresses around the elliptical hole.

The stress analysis has the flexibility to include thermal stresses and also to evaluate energy release rates $\mathcal{G}$. This can be done by solving the problems with different flaw lengths and evaluating $\mathcal{G}$ from the change in stored energy of the system per unit area of extension in flaw. The procedures for doing the thermal stress analysis and the energy release rate calculations are outlined in Appendices C, D, and E. Appendix F contains description of the input data for the computer program.

SECTION III
RESULTS

Several problems were run wihh the equilibrium finite element stress analysis program. These problems involve a four-ply $(90/0)_s$ laminate. The configurations and loading conditions were a laminate containing a circular hole under inplane loading and a laminate containing an elliptical hole also with inplane loading.

The principal material properties for the plys are

$$E_1 = 30. \qquad\qquad E_2 = E_3 = 3.0$$

$$\nu_{13} = \nu_{12} = \nu_{23} = 0.336$$

and

$$G_{12} = G_{13} = G_{23} = 1.$$

The radius of the hole was 1.0. The thickness of each ply was 0.2 inch. Stress boundary and conditions were applied at a radius of 12 inches from the center of the hole. These stresses were evaluated from a laminated plate theory representation for a uniform resultant stress in one direction.

The first configuration was a $(90/0)_s$ laminate containing a circular hole and inplane loading in the x-direction. A reference solution for this problem was obtained by three-dimensional displacement finite-element computer program called SAP IV. The obtained solution and the reference solution for the interlaminar stress, σ_z, around the circular free edge are shown in Figure 2. Figure 3 shows the σ_z distributions along the x and y directions for the equilibrium finite-element analysis.

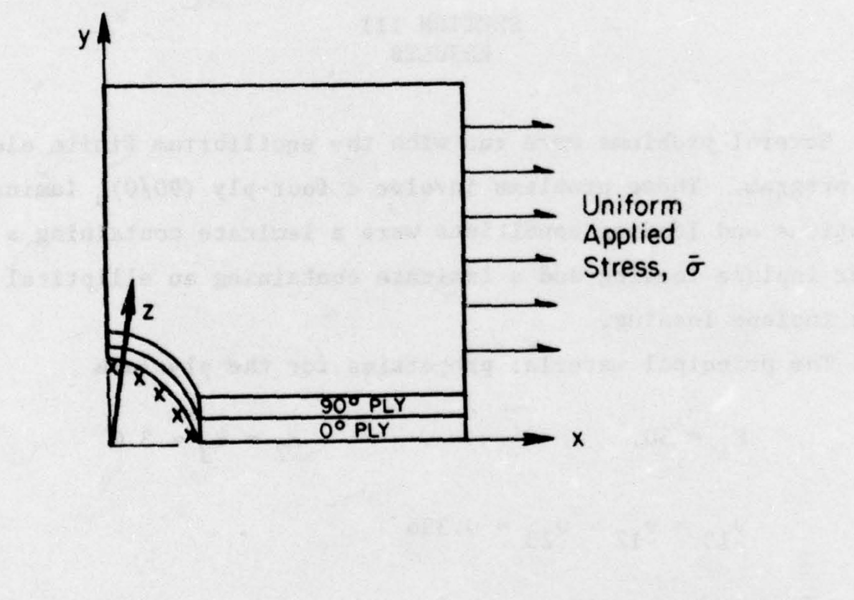

FIGURE 2. DISTRIBUTION OF INTERLAMINAR STRESS, σ_z, AROUND A CIRCULAR HOLE IN A $(90/0)_s$ LAMINATE

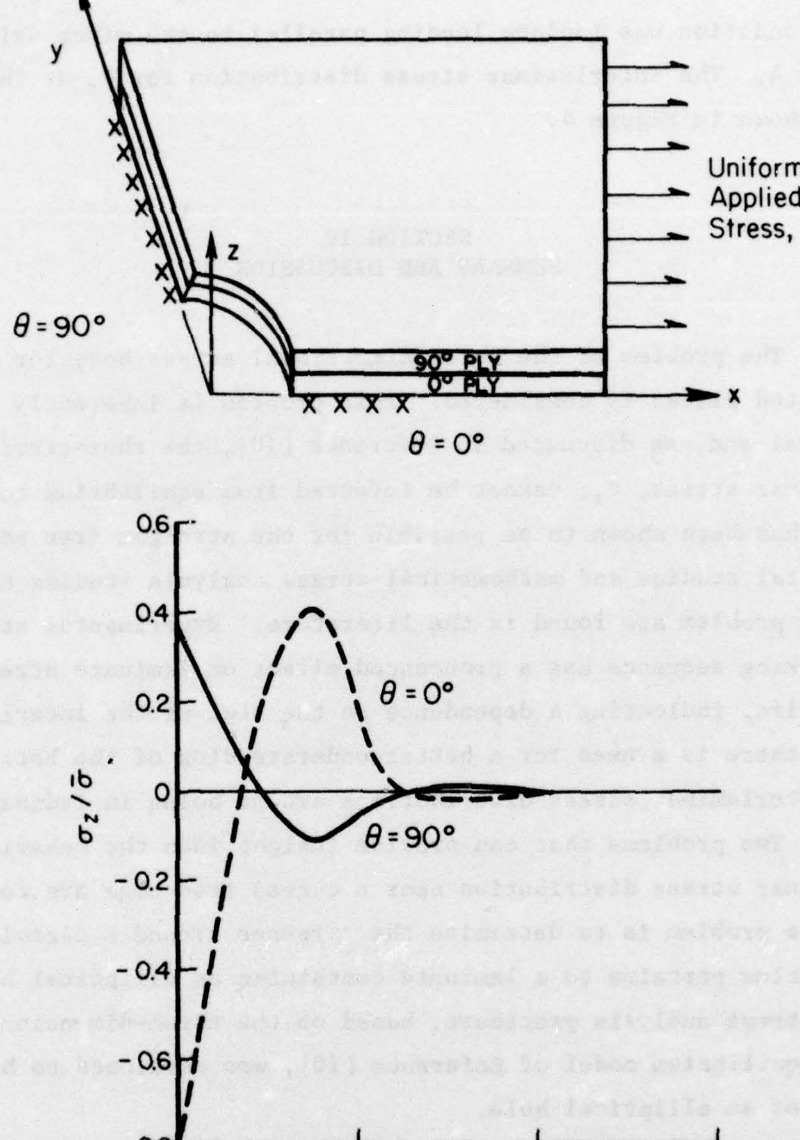

FIGURE 3. DISTRIBUTIONS OF INTERLAMINAR STRESS, σ_z, ALONG $\theta = 0$ AND $\theta = 90°$ FOR $(90/0)_s$ LAMINATE CONTAINING A CIRCULAR HOLE

Next the problem of an elliptical hole in $(90/0)_2$ laminate was considered. The ratio of the minor axis to the major axis was 0.5. The loading condition was inplane loading parallel to the minor axis as shown in Figure 4. The interlaminar stress distribution for σ_z at the midsurface is also shown in Figure 4.

SECTION IV
SUMMARY AND DISCUSSION

The problem of the three-dimensional stress behavior around holes in laminated plates is considered. This problem is inherently three-dimensional and, as discussed in Reference [10], the characteristics of the interlaminar stress, σ_z, cannot be inferred from equilibrium considerations alone as has been shown to be possible for the straight free edge problem. Experimental studies and mathematical stress analysis studies addressing the free edge problem are found in the literature. Experimental studies show that stacking sequence has a pronounced effect on laminate strength and fatigue life, indicating a dependence on the sign of the interlaminar stresses. However, there is a need for a better understanding of the basic behavior of the interlaminar stress distributions around holes in laminated plates.

Two problems that can provide insight into the behavior of the interlaminar stress distribution near a curved free edge are considered here. One problem is to determine the stresses around a circular hole. The other problem pertains to a laminate containing an elliptical hole. A mathematical stress analysis procedure, based on the three-dimensional finite element equilibrium model of Reference [10], was developed to handle the problems of an elliptical hole.

The first problem was that of a circular hole in a $(90/0)_s$ laminate loaded with a uniform inplane extension in the direction of the x-axis. Here a reference solution was available. The reference solution was obtained by a three-dimensional compatible finite element stress analysis called SAP IV. The comparison of the interlaminar stress distributions around the circular hole was good. The equilibrium element stress analysis gave higher stress gradients and higher peak values of σ_z, but both analyses revealed a changing sign of σ_z around the circular edge.

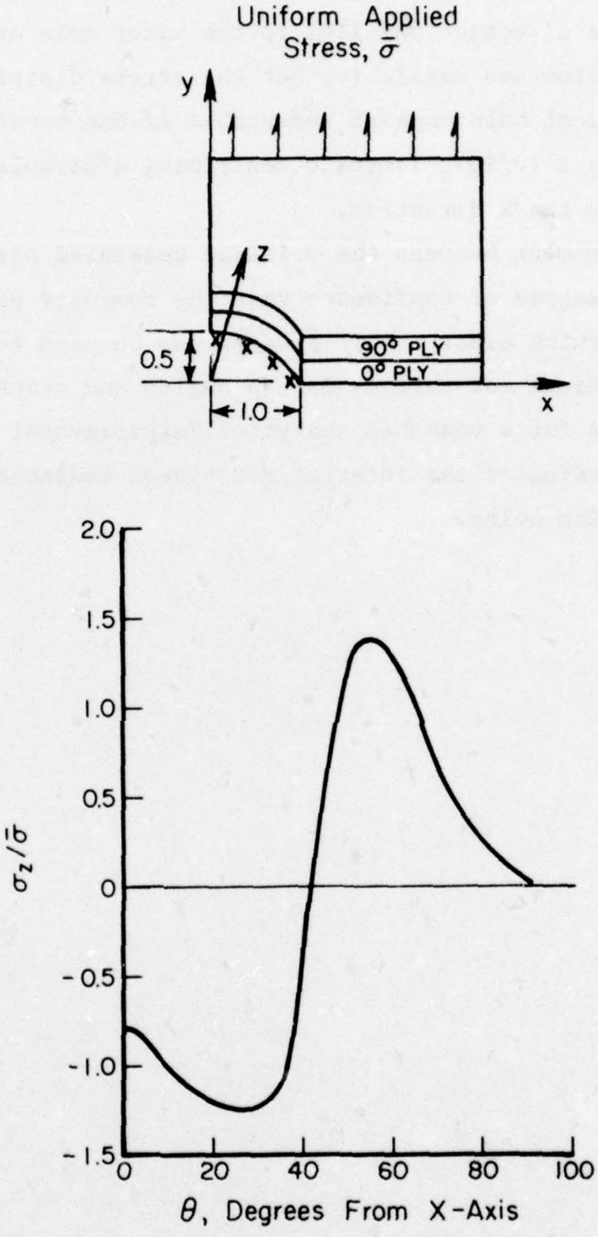

FIGURE 4. DISTRIBUTION OF INTERLAMINAR STRESS, σ_z, AROUND AN ELLIPTICAL HOLE IN A $(90/0)_s$ LAMINATE

The next problem was that of $(90/0)_s$ laminated plate containing an elliptical hole with a ratic of minor axis to major axis of 1/2. The loading was in the direction parallel to the minor axis or in the y direction. No reference solution was available, but the stress distribution for σ_z around the elliptical hole appears reasonable if one considers this case is similar to loading a $(0/90)_s$ laminate containing a circular hole with inplane loading in the x direction.

The agreement between the solution generated here and the reference solution gives a degree of confidence that the computer program for the analysis is in working order. This program can be used to generate a catalogue of solutions for various lay-up angles and stacking sequences to provide a basis for a combined analytical/experimental program to obtain a better understanding of the interlaminar stress and strain behavior in laminates containing holes.

APPENDIX A

TRANSFORMATION OF BOUNDARY CONDITIONS BETWEEN THE CIRCULAR AND ELLIPTICAL HOLE BOUNDARY

The transformation of stress boundary conditions between the circular and elliptical hole problems can be represented by

$$\{\sigma^E\}_{N\theta Z} = [C] \{\sigma^C\}_{N\theta Z} \tag{A-1}$$

where

$$\{\sigma^E\}_{N\theta Z} = \begin{Bmatrix} \sigma_N^E \\ \sigma_\theta^E \\ \sigma_Z^E \\ \tau_{N\theta}^E \\ \tau_{NZ}^E \\ \tau_{\theta Z}^E \end{Bmatrix} \quad ; \quad \{\sigma^C\}_{N\theta Z} = \begin{Bmatrix} \sigma_N^C \\ \sigma_\theta^C \\ \sigma_Z^C \\ \tau_{N\theta}^C \\ \tau_{NZ}^C \\ \tau_{\theta Z}^C \end{Bmatrix}$$

and [C] is a transformation matrix.

Let [T] represent the transformation matrix which relates stresses in polar coordinates to stresses in cartesian coordinates. For the circular hole,

$$\{\sigma^C\}_{N\theta Z} = \{T^C\} \{\sigma^C\}_{XYZ} \quad , \tag{A-2}$$

and for the elliptical hole,

$$\{\sigma^E\}_{N\theta Z} = \{T^C\} \{\sigma^E\}_{XYZ} \quad . \tag{A-3}$$

The matrix [C] in Equation (A-1) can be derived from Equation (A-1), Equation A-3, and the transformations which relate cartesian stresses of the circular hole problem to cartesian stresses of the elliptical hole problem. These transformations are

$$\sigma_X^E = \sigma_X^C/\beta^2 \qquad \tau_{XY}^E = \tau_{XY}^C/\beta$$

$$\sigma_Y^E = \sigma_Y^C \qquad \tau_{XZ}^C = \tau_{XZ}^C/\beta^2 \qquad \text{(A-4)}$$

$$\sigma_Z^E = \sigma_Z^C/\beta^2 \qquad \tau_{YZ}^E = \tau_{YZ}^C$$

where β is defined by the cartesian coordinate transformation,

$$\begin{aligned} X^E &= X^C \\ Y^E &= \beta Y^E \\ Z^E &= Z^C \end{aligned} \qquad \text{(A-5)}$$

From Equation (A-2),

$$\{\sigma^C\}_{XYZ} = [T^C]^{-1}\{\sigma^C\}_{Y\theta Z} \qquad \text{(A-6)}$$

where

$$[T^{C'}]^{-1} = \begin{matrix} T_1^C/\beta^2 \\ T_2^C \\ T_3^C/\beta^2 \\ T_4^C/\beta \\ T_5^C/\beta^2 \\ T_6/\beta \end{matrix} \qquad \text{(A-7)}$$

and T_i^C denotes the i^{th} row of $[T^C]^{-1}$.

Using Equation (A-6) and Equation (A-3),

$$\{\sigma^E\}_{N\theta Z} = [T^E]\,\{\sigma^E\}_{XYZ} = [T^E][T^{C'}]^{-1}\{\sigma^C\}_{N\theta Z} \quad . \qquad \text{(A-8)}$$

Thus,

$$[C] = [T^E][T^{C'}]^{-1} \quad .$$

From Equation (A-8) it can be shown that the boundary conditions for the elliptical hole problem are $\sigma_N^E(\overline{r},\theta,Z) = \tau_{NZ}^E(\overline{r},\theta,Z) = \tau_{N\theta}^E(\overline{r},\theta,Z) = 0$ when the boundary conditions for the circular hole problem are $\sigma_N^C(\overline{r},\theta,Z) = \tau_{N\theta}^C(\overline{r},\theta,Z) = \tau_{NZ}^C(\overline{r},\theta,Z) = 0$

APPENDIX B

SATISFYING STRESS BOUNDARY CONDITIONS BY PRESCRIBING APPROPRIATE $F_{ijk}^{(I)}$ VALUES

The three-dimensional finite element model used in the analysis of the laminated plate is shown in Figure B-1. The stress boundary conditions applied to this model are listed below.

Table B-1

Location	Boundary Conditions
$r = r_0$	$\sigma_r(r_0,\theta,Z) = \tau_{r\theta}(r_0,\theta,Z) = \tau_{rZ}(r_0,\theta,Z) = 0$
$r = r_S$	Uniform Stress Field
$Z = 0$	$\tau_{\theta Z}(r,\theta,Z_0) = \tau_{\theta r}(r,\theta,Z_0) = 0$
$Z = Z_S$	$\sigma_Z(r,\theta,Z_S) = \tau_{\theta Z}(r,\theta,Z_S) = \tau_{\theta r}(r,\theta,Z_S) = 0$

The following sections describe how these boundary conditions are met by prescribing appropriate $F_{ijk}^{(I)}$ coefficients which appear in the expressions for the stress functions, χ_I. These stress functions have the form

$$\chi_I = \sum_i \sum_j \sum_k F_{ijk}^{(I)} P_i(r) Q_j(\theta) R_k(Z)$$

where I = 1, 2, or 3.

Boundary Conditions at $R = R_0$

On the inner hole surface, $P_1 = P_3' = 1$, while all other P_i's are zero. The expressions for $\sigma_r(r_0)$, $\tau_{r\theta}(r_0)$, and $\tau_{rZ}(r_0)$ and therefore

$$\sigma_r(r_0) = \sum_j \sum_k \Big\{ F_{ijk}^{(1)} Q_j R_k''(1-\cos^2\theta) + F_{ijk}^{(2)} Q_j R_k'' \cos^2\theta + + F_{ijk}^{(3)} (Q_j''/r^2) R_k + F_{3jk}^{(3)} (Q_j/r) R_k \Big\} = 0 \tag{B-1}$$

$$\tau_{r\theta}(r_0) = \sum_j \sum_k \left\{ F^{(1)}_{ijk} Q_j R''_k \cos\theta \cdot \sin\theta - F^{(2)}_{ijk} Q_j R''_k \cos\theta \sin\theta + \right.$$

$$\left. - F^{(3)}_{3jk}(Q'_j/r)R_k + F^{(3)}_{1jk}(Q'_j/r^2)R_k \right\} = 0 \quad , \tag{B-2}$$

$$\tau_{rZ}(r_0) = \sum_j \sum_k \left\{ -F^{(1)}_{3jk} Q_j R'_k \sin^2\theta - F^{(1)}_{1jk}(Q'_j/r)R'_k \cos\theta \cdot \sin\theta + \right.$$

$$\left. + F^{(2)}_{1jk}(Q'_j/r)R'_k \cos\theta \cdot \sin\theta - F^{(2)}_{3jk} Q_j R'_k \cos^2\theta \right\} = 0 \quad . \tag{B-3}$$

Setting the terms containing $F^{(3)}_{ijk}$ in Equations (B-1) and (B-2) to zero gives

$$\sum_j \sum_k F^{(3)}_{1jk}(Q''_j/r^2)R_k + \sum_j \sum_k F^{(3)}_{3jk}(Q_j/r)R_k = 0 \quad , \tag{B-4}$$

$$-\sum_j \sum_k F^{(3)}_{3jk}(Q'_j/r)R_k + F^{(3)}_{1jk}(Q'_j/r^2)R_k = 0 \quad . \tag{B-5}$$

Two sets of constraint equations are generated by using symmetric and antisymmetric Q_j terms, respectively. For example, the expansion using symmetric terms $y = 1,3,5,\ldots,11$ is

$$\left\{ F^{(1)}_{33k} + F^{(2)}_{31k} \right\} \cos^2\theta + \left\{ -F^{(1)}_{33k} + F^{(1)}_{35k} + F^{(2)}_{33k} \right\} \cos^4\theta +$$

$$+ \left\{ -F^{(1)}_{35k} + F^{(1)}_{37k} + F^{(2)}_{35k} \right\} \cos^6\theta + \left\{ -F^{(1)}_{37k} + F^{(1)}_{39k} + F^{(2)}_{37k} \right\} \cos^8\theta + \tag{B-6}$$

$$+ \left\{ -F^{(1)}_{39k} + F^{(1)}_{3,11,k} + F^{(2)}_{39k} \right\} \cos^{10}\theta + \left\{ -F^{(1)}_{3,11,k} + F^{(2)}_{3,11,k} \right\} \cos^{12}\theta = 0$$

$$k = 1,2,3,4 \quad .$$

Therefore, the set of constraint equations is

$$F^{(1)}_{33k} + F^{(2)}_{31k} = 0 \quad ,$$

$$-F^{(1)}_{33k} + F^{(1)}_{35k} + F^{(2)}_{33k} = 0 \quad ,$$

$$-F^{(1)}_{35k} + F^{(1)}_{37k} + F^{(2)}_{35k} = 0 \quad ,$$

$$-F^{(1)}_{37k} + F^{(1)}_{39k} + F^{(2)}_{37k} = 0 \quad , \qquad \text{(B-7)}$$

$$-F^{(1)}_{39k} + F^{(1)}_{3,11,k} + F^{(2)}_{39k} = 0 \quad ,$$

$$-F^{(1)}_{3,11,k} + F^{(2)}_{3,11,k} = 0 \quad , \qquad k = 1,2,3,4 \quad .$$

The expansion using antisymmetric terms y = 2,4,6,...10 gives the owing set of constraint equations.

$$-F^{(1)}_{34k} + F^{(1)}_{36k} + F^{(2)}_{34k} = 0 \quad ,$$

$$-F^{(1)}_{36k} + F^{(1)}_{38k} + F^{(2)}_{36k} = 0 \quad , \qquad \text{(B-8)}$$

$$-F^{(1)}_{38k} + F^{(1)}_{3,10,k} + F^{(2)}_{3,18,k} = 0 \quad ,$$

$$-F^{(1)}_{3,10,k} + F^{(2)}_{3,10,k} = 0 \quad , \qquad k = 1,2,3,4 \quad .$$

<u>Boundary Conditions at $R = R_t$</u>

The following conditions satisfy the uniform stress boundary conditions at $R = R_t$.

$$\sigma_x = \bar{\sigma}_x \ , \quad \sigma_y = \bar{\sigma}_y \quad \text{and} \quad \tau_{xy} = \bar{\tau}_{xy}$$

where $\bar{\sigma}_x$, $\bar{\sigma}_y$, and $\bar{\tau}_{xy}$ are prescribed. By letting

$$\chi_3 = \frac{1}{2}\,\bar{\sigma}_x R_F^2 \sin^2\theta + \frac{1}{2}\,\bar{\sigma}_y R_F^2 \cos^2\theta - \bar{\tau}_{xy} F_F^2 \cos\theta \sin\theta \quad ,$$

the uniform stress requirement at $R = R_F$ is met by prescribing the following $F^{(3)}_{ijk}$ values.

$$F^{(3)}_{211} = F^{(3)}_{212} = \frac{1}{2}\,\bar{\sigma}_x R_F^2 \qquad F^{(3)}_{411} = F^{(3)}_{412} = \bar{\sigma}_x R_F$$

$$F^{(3)}_{231} = F^{(3)}_{232} = \frac{1}{2}\,R_F^2(\bar{\sigma}_y - \bar{\sigma}_x) \qquad F^{(3)}_{431} = F^{(3)}_{432} = R_F(\bar{\sigma}_y - \bar{\sigma}_x) \tag{B-9}$$

$$F^{(3)}_{221} = F^{(3)}_{222} = -\bar{\tau}_{xy} R_F^2 \qquad F^{(3)}_{421} = F^{(3)}_{422} = -2\,\bar{\tau}_{xy}\,R_F$$

$$F^{(3)}_{2jk} = F^{(3)}_{ijk} = 0 \qquad \text{for } k = 3,4$$

The prescribed values of $F^{(1)}_{ijk}$ and $F^{(2)}_{ijk}$ are obtained by the same analysis that was applied at the inner radius, except that at $R = R_t$, $P_2 = P_u = 1$, while all other P_i's and P_i''s are zero. At $R = R_t$ the analysis gives the following prescribed F^{I}_{ijk} values and constraint equations.

$$F^{(1)}_{ijk} = F^{(2)}_{2jk} = 0 \qquad \text{for all } j \text{ and } k \ ,$$

$$F^{(1)}_{43k} + F^{(2)}_{41k} = 0 \ , \qquad FF^{(1)}_{44k} + F^{(2)}_{42k} = 0 \ ,$$

$$-F^{(1)}_{43k} + F^{(1)}_{45k} + F^{(2)}_{43k} = 0 \ , \qquad -F^{(1)}_{44k} + F^{(1)}_{46k} + F^{(2)}_{44k} = 0 \ ,$$

$$-F^{(1)}_{45k} + F^{(1)}_{47k} + F^{(2)}_{45k} = 0 \ , \qquad -F^{(1)}_{46k} + F^{(1)}_{48k} + F^{(2)}_{46k} = 0 \ , \tag{B-10}$$

$$-F^{(1)}_{47k} + F^{(1)}_{49k} + F^{(2)}_{47k} = 0 \ , \qquad -F^{(1)}_{48k} + F^{(1)}_{4,10,k} + F^{(2)}_{48k} = 0 \ ,$$

$$-F^{(1)}_{49k} + F^{(1)}_{4,11,k} + F^{(2)}_{49k} = 0 \ , \qquad -F^{(1)}_{4,10,k} + F^{(2)}_{4,10,k} = 0 \ ,$$

$$-F^{(1)}_{4,11,k} + F^{(2)}_{4,11,k} = 0 \ , \qquad k = 1,2,3,4 \ .$$

These equations and prescribed F^{I}_{ijk} values apply to elements 2,4, and 6 of the finite element model.

Boundary Conditions at $Z = 0$

On the lower plate surface, $R_2 = R_3' = 1$ while all other R_k's and R_k''s are zero. These conditions and the relationships

$$P_1' = -P_2' \;, \quad P_1' = -P_2'' \;, \quad Q_1' = Q_1''' = 0 \;, \tag{B-11}$$

give the following prescribed conditions for elements 1 and 2.

$$\left.\begin{array}{ll} F_{113}^{(1)} = F_{213}^{(1)} & F_{ij3}^{(1)} = 0 \\ F_{113}^{(2)} = F_{213}^{(2)} & F_{ij3}^{(2)} = 0 \end{array}\right\} \quad j > 1 \;, \quad i = 3,4,5,6 \tag{B-12}$$

Boundary Conditions at $Z = Z_S$

On the upper plate surface, $R_2 = R_4' = 1$ while all other R_k's and R_k''s are zero. These conditions and Equation (B-11) give the following prescribed conditions for elements 5 and 6.

$$\begin{array}{llll} F_{112}^{(1)} = F_{212}^{(1)} \;, & F_{ij2}^{(1)} = 0 \;, & j > 1 \;, & i = 3,4,5,6 \\ F_{114}^{(1)} = F_{214}^{(1)} \;, & F_{ij4}^{(1)} = 0 \;, & j > 1 \;, & i = 3,4,5,6 \\ F_{112}^{(2)} = F_{212}^{(2)} \;, & F_{ij2}^{(2)} = 0 \;, & j > 1 \;, & i = 3,4,5,6 \\ F_{114}^{(2)} = F_{214}^{(2)} \;, & F_{ij4}^{(2)} = 0 \;, & j > 1 \;, & i = 3,4,5,6 \end{array} \tag{B-13}$$

A comparison of the appropriate equations developed in this appendix shows the specified F_{ijk}^{I} values to be compatible in elements which are bounded by more than one surface.

APPENDIX C

THERMAL STRESS ANALYSIS

This appendix describes how thermal stresses due to a constant uniform temperature were included in the laminated plate analysis. The procedure is listed in terms of the following three steps.

Step 1. Write the expression for the part of the complementary energy density, u_c^T, due to thermal strains and obtain an expression for the thermal strains in the (r,θ,z) reference coordinate system.

Step 2. Express u_c^T in terms of the stress functions and the thermal strains of Step 1.

Step 3. List the integrals of r, θ, and z that are required and determine how to evaluate them.

In Step 1, the expression for the complementary energy density due to the thermal strains is denoted by u_c^T and can be written as

$$u_c^T = \epsilon_r^T \sigma_r + \epsilon_\theta^T \sigma_\theta + \epsilon_z^T \sigma_z + \gamma_{rz}^T \tau_{rz} + \gamma_{\theta z}^T \tau_{\theta z} + \gamma_{r\theta}^T \tau_{r\theta} \tag{C-1}$$

The material is assumed to have three principal coefficients of thermal expansion that coincide with the principal material directions. These are denoted by α_1, α_2, and α_3. Figure 1 shows the principal material directions, the (x,y,z) and (r,θ,z) coordinate systems for a typical lamina. Expressions for the thermal strains in the (r,θ,z) systems are needed for the analysis. To obtain these expressions, the thermal strains are first transformed from the $(1,2,3)$ system to the (x,y,z) system and then to the (r,θ,z) system.

Transforming the thermal strains from the $(1,2,3)$ system to the (x,y,z) system gives

$$\epsilon_x^T = (\alpha_1 \cos^2\phi + \alpha_2 \sin^2\phi)\,\Delta T \quad , \tag{C-2}$$

$$\epsilon_y^T = (\alpha_1 \sin^2\phi + \alpha_2 \cos^2\phi)\,\Delta T \quad , \tag{C-3}$$

$$\epsilon_z^T = \alpha_3 \Delta T \quad , \tag{C-4}$$

$$\gamma_{xy}^T = (2\alpha_1 \cos\phi \sin\phi - 2\alpha_2 \cos\phi \sin\phi)\ \Delta T \quad , \tag{C-5}$$

and $$\gamma_{xz}^T = \gamma_{yz}^T = 0 \quad , \tag{C-6}$$

where ϕ is shown in Figure 1 and ΔT is the temperature change. Since ϕ, ΔT, α_1, α_2, and α_3 are constants for any lamina, the strains in Equations (C-2) through (C-6) are also constants for any lamina.

The result of transforming Equations (C-2) through (C-6) to the (r,θ,z) directions is

$$\epsilon_r^T = \epsilon_x^T m^2 + \epsilon_y^T n^2 + \gamma_{xy}^T mn \quad , \tag{C-7}$$

$$\epsilon_\theta^T = \epsilon_x^T n^2 + \epsilon_y^T m^2 - \gamma_{xy}^T mn \quad , \tag{C-8}$$

$$\epsilon_z^T = \epsilon_z^T \quad , \tag{C-9}$$

$$\gamma_{r\theta}^T = -\ 2\epsilon_x^T mn + 2\epsilon_y^T mn + (m^2 - n^2)\ \gamma_{xy}^T \quad , \tag{C-10}$$

and $$\gamma_{rz}^T = \gamma_{\theta z}^T = 0 \quad , \tag{C-11}$$

where $$m = \cos\theta \tag{C-12}$$

and $$n = \sin\theta \quad . \tag{C-13}$$

The strains in Equations (C-7) through (C-11) are functions of θ. Combining Equations (C-7) through (C-11) with Equation (C-1) gives an expression for the complementary energy due to the thermal strains in terms of stresses and strains in the polar (n,θ,z) coordinate system.

In Step 2, the stresses that are needed to evaluate u_c^T in Equation (C-1) are σ_r, σ_θ, σ_z and $\tau_{r\theta}$. The stresses τ_{rz} and $\tau_{\theta z}$ are not needed because $\gamma_{rz}^T = \gamma_{\theta z}^T = 0$ from Equation (C-11). The expression for u_c^T in terms of the stress functions and the thermal strains is obtained by expressing the stresses

in Equation (C-1) in terms of the stress functions and substituting Equations (C-7) through (C-13) into Equation (C-1). The result for a lamina is

$$u_c^T = (\epsilon_x^T \cos^2\theta + \epsilon_y^T \sin^2\theta + \gamma_{xy}^T \cos\theta \sin\theta) \times$$

$$\sum_{ijk}\sum\sum \Big\{ F_{ijk}^{(1)} P_i(r)\, Q_j(\theta)\, R_k''(z) +$$

$$- F_{ijk}^{(1)} P_i(r)\, Q_j(\theta)\, R_k''(z) \cos^2\theta +$$

$$+ F_{ijk}^{(2)} P_i(r)\, Q_j(\theta)\, R_k''(z) \cos^2\theta +$$

$$+ F_{ijk}^{(3)} (P_i(r)/r^2) Q_j''(\theta)\, R_k(z) +$$

$$+ F_{ijk}^{(3)} (P_i'(r)/r)\, Q_j(\theta)\, R_k(z) \Big\} +$$

$$+ \left(\epsilon_x^T \sin^2\theta + \epsilon_y^T \cos^2\theta - \gamma_{xy}^T \cos\theta \sin\theta\right) \times$$

$$\sum_{ijk}\sum\sum \Big\{ F_{ijk}^{(1)} P_i(r)\, Q_j(\theta)\, R_k''(z) \cos^2\theta +$$

$$+ F_{ijk}^{(2)} P_i(r)\, Q_j(\theta)\, R_k''(z) +$$

$$- F_{ijk}^{(2)} P_i(r)\, Q_j(\theta)\, R_k''(z) \cos^2\theta +$$

$$+ F_{ijk}^{(3)} P_i''(r)\, Q_j(\theta)\, R_k(z) \Big\} + \epsilon_z^T \times$$

$$\sum_{ijk}\sum\sum \Big\{ F_{ijk}^{(1)} P_i''(r)\, Q_j(\theta)\, R_k(z) +$$

$$- F_{ijk}^{(1)} P_i''(r)\, Q_j(\theta)\, R_k(z) \cos^2\theta +$$

$$+ 2F^{(1)}_{ijk} \; (P'_i(r)/r) \; Q'_j(\theta) \; R_k(z) \; \cos\theta \sin\theta \; +$$

$$+ F^{(1)}_{ijk} \; (P_i(r)/r^2) \; Q''_j(\theta) \; R_k(z) \; \cos^2\theta \; +$$

$$+ F^{(1)}_{ijk} \; (P'_i(r)/r) \; Q_j(\theta) \; R_k(z) \; \cos^2\theta \; +$$

$$- 2F^{(1)}_{ijk} \; (P_i(r)/r^2) \; Q'_j(\theta) \; R_k(z) \; \cos\theta \sin\theta \; +$$

$$+ F^{(2)}_{ijk} \; P''_i(r) \; Q_j(\theta) \; R_k(z) \; \cos^2\theta \; +$$

$$- 2F^{(2)}_{ijk} \; (P'_i(r)/r) \; Q'_j(\theta) \; R_k(z) \; \cos\theta \sin\theta \; +$$

$$+ F^{(2)}_{ijk} \; (P_i(r)/r^2) \; Q''_j(\theta) \; R_k(z) \; +$$

$$- F^{(2)}_{ijk} \; (P_i(r)/r^2) \; Q''_j(\theta) \; R_k(z) \; \cos^2\theta \; +$$

$$+ F^{(2)}_{ijk} \; (P'_i(r)/r) \; Q_j(\theta) \; R_k(z) \; +$$

$$- F^{(2)}_{ijk} \; (P'_i(r)/r) \; Q_j(\theta) \; R_k(z) \; \cos^2\theta \; +$$

$$+ 2F^{(2)}_{ijk} \; (P_i(r)/r^2) \; Q'_j(\theta) \; R_k(z) \; \cos\theta \sin\theta \Big\} \; +$$

$$+ \left(-2\epsilon^T_x \cos\theta \sin\theta + 2\epsilon^T_y \cos\theta \sin\theta - (2\cos^2\theta - 1)\gamma^T_{xy}\right) \; x$$

$$\sum_{ijk}\sum\sum \Big\{ F^{(1)}_{ijk} \; P_i(r) \; Q_j(\theta) \; R''_k(z) \; \cos\theta \sin\theta \; +$$

$$- F^{(2)}_{ijk} \; P_i(r) \; Q_j(\theta) \; R''_k(z) \; \cos\theta \sin\theta \; +$$

$$- F^{(3)}_{ijk} \; (P'_i(r)/r) \; Q'_j(\theta) \; R_k(z) \; +$$

$$+ F^{(3)}_{ijk} \; (P_i(r)/r^2) \; Q'_j(\theta) \; R_k(z) \Big\} \qquad \text{(C-14)}$$

In Equation (C-14), $P_i(r)$, $Q_j(\theta)$, and $R_k(z)$ are the assumed functions for the finite element representation of the three stress functions. The $F_{ijk}^{(L)}$'s are the unknown coefficients for the stress functions associated with $P_i(r)$, $Q_j(\theta)$, $R_k(z)$ and the Lth stress function. In solving thermal stress problems, the value of $\frac{\partial}{\partial F_{ijk}^{(L)}} \left\{ \iiint_V u_c^T dv \right\}$ must be evaluated and

subtracted from the right-hand side of the system of equations. It can be seen from Equation (C-14) that the following integrals are needed to evaluate these terms.

$$\int_{r_1}^{r_2} P_i(r) r dr \ ,$$

$$\int_{r_1}^{r_2} \frac{P_i(r)}{r^2} r dr = \int_{r_1}^{r_2} \frac{P_i(r)}{r} dr \ ,$$

$$\int_{r_1}^{r_2} \frac{P_i(r)}{r} r dr = P_i(r_2) - P_i(r_1) \ ,$$

$$\int_{r_1}^{r_2} P_i''(r) r dr = r_2 P_i'(r_2) - r_1 P_i'(r_1) - P_i(r_2) + P_i(r_1) \ ,$$

$$\int_{\theta_1}^{\theta_2} Q_j(\theta) f(\theta) d\theta \ ,$$

where $f(\theta) = 1, \cos^2\theta, \sin^2\theta, \cos^4\theta, \cos^2\theta \sin^2\theta,$
$\cos\theta \sin\theta, \cos^3\theta \sin\theta,$

$$\int_{\theta_1}^{\theta_2} Q_j'(\theta) g(\theta) d\theta \ ,$$

where $g(\theta) = 1, \cos\theta \sin\theta, \cos^2\theta, \sin^2\theta,$

$$\int_{\theta_1}^{\theta_2} Q_j''(\theta)h(\theta)d\theta ,$$

where $h(\theta) = 1, \cos\theta\sin\theta, \cos^2\theta, \sin^2\theta$,

$$\int_{z_1}^{z_2} R_k(z)dz ,$$

and $$\int_{z_1}^{z_2} R_k''(z)dz = R_k'(z_2) - R_k'(z_1) .$$

The integrals in r can be evaluated by using the existing procedure for $\int P_i(r)\ P_\ell(r)f(r)rdr$ and setting $P_\ell(r) = 1$, $f(r) = 1$. Values for the integrals of θ can be obtained with the existing procedure for evaluating $\int D_p Q_j(\theta)Q_m(\theta)f(\theta)d\theta$ by setting $Q_m(\theta) = 1$, $f(\theta)$ equal to the expressions listed above for $f(\theta)$, $g(\theta)$, and $h(\theta)$, and setting $D_p = 1, \frac{d}{d\theta}, \frac{d^2}{d\theta^2}$ as needed. Similarly, values for the z-integrals can be obtained with the existing procedure for $\int R_k(z)R_n(z)dz$ with $R_n(z) = 1$.

The effect of the thermal strains is to change the right-hand side of the system of equations to be solved by subtracting the vector of values given by $\partial/\partial F_{ijk}^{(L)}\{\iiint_V u_c^T dV\}$. A subroutine to evaluate the thermal contribution to the right-hand side of the equations was written and added to the program.

APPENDIX D

THROUGH CRACK PROBLEM

An extension to the analysis was made to handle the problem of a through crack from a cricular hole. Two cracks of equal length, along a diameter were considered. The approach taken here is to add the stress-free boundary conditions for the crack faces and compute an energy release rate for the configuration. The requirements for stress-free boundary conditions are given in Appendix E. The energy release rate, $\mathcal{G}$, is the amount of energy per unit of length that the laminated plate would give up if the crack were to extend a small amount. The expression for $\mathcal{G}$ is

$$\mathcal{G} = \lim_{\Delta C \to 0} \frac{E(C+\Delta C)-E(C)}{t\Delta C} \tag{D-1}$$

where C is the crack length, ΔC is an assumed change in crack length, $E(C+\Delta C)$ and $E(C)$ are the strain energies for the laminate with crack lengths $C+\Delta C$ and C, respectively, and t is the laminate thickness.

For linear elastic materials, the strain energy is equal to the complementary energy of the laminate. The expression for the complementary energy of the laminate can be written as

$$\pi_c = \frac{1}{2}\,[F, F_p] \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{Bmatrix} F \\ F_p \end{Bmatrix} \tag{D-2}$$

where F is the vector of unknown coefficients, F_p is the vector of prescribed coefficients, and $[C_{ij}]$ is the symmetric flexibility matrix. Multiplying out Equation (D-2) gives

$$\pi_c = \frac{1}{2}\left[F^T C_{11} F + 2F^T C_{12} F_p + F_p^T C_{22} F_p^T\right] \tag{D-3}$$

The solution of the problem, F*, is obtained from the equation

$$\nabla_F \pi_C = C_{11} F^* + C_{12} F_p = 0 \tag{D-4}$$

In Equation (D-4) the term $C_{12}F_p$ is the right-hand side, RHS, of the equations to be solved. Thus, from Equation (D-4)

$$C_{11}F^* = -RHS \tag{D-5}$$

Substituting Equation (D-5) into Equation (D-3) and putting the solution F* for F gives

$$\pi_C = \frac{1}{2}\left[F^{*T}(-RHS) + 2F^{\,T}(RHS) + F_p^TC_{22}F_p^T\right]$$

or

$$\pi_C = \frac{1}{2}\left[F^{*T}RHS + F_p^TC_{22}F_p\right] \tag{D-6}$$

The first term in Equation (D-6) is the vector product of the solution vector and the right-hand side vector. The second term in Equation (25) is contribution due to the prescribed coefficients F_p. This term can be evaluated before the solution vector F* is solved for.

Since the strain energy E is equal to the complementary energy, π_c, Equation (D-2) can be written as

$$\mathcal{G} = \lim_{\Delta C \to 0} \frac{\pi_C(C+\Delta C)-\pi_C(C)}{t\Delta C}$$

This procedure for evaluating $\mathcal{G}$ was included in the program.

A description of the computer program and the input data is given in Appendix F.

APPENDIX E

FORMULATION OF SEVEN CRACK PROPAGATION PROBLEMS BY IMPOSING ADDITIONAL STRESS BOUNDARY CONDITIONS ON ELEMENTS 1, 3, AND 5

A crack originating at the inner plate radius is included in the finite element analysis by applying the boundary conditions

$$\sigma_{\theta}\Big|_{\substack{\theta = 0^\circ \\ \theta = 180^\circ}} = \tau_{r\theta}\Big|_{\substack{\theta = 0^\circ \\ \theta = 180^\circ}} = \tau_{\theta z}\Big|_{\substack{\theta = 0^\circ \\ \theta = 180^\circ}} = 0$$

to elements at the crack location. For each of these elements, the above boundary conditions reduce to

$$\sum_i \sum_j \sum_k \left\{ F^{(1)}_{ijk} P_i Q_j R''_k + F^{(3)}_{ijk} P''_i Q_j R_k \right\} \Bigg|_{\substack{\theta = 0^\circ \\ \theta = 180^\circ}} = 0 \qquad \text{(E-1)}$$

$$\sum_i \sum_j \sum_k \left\{ -F^{(3)}_{ijk} (P'_i/r) Q'_j R_k + F^{(3)}_{ijk} (P_i/r^2) Q'_j R_k \right\} \Bigg|_{\substack{\theta = 0^\circ \\ \theta = 180^\circ}} = 0 \qquad \text{(E-2)}$$

$$\sum_i \sum_j \sum_k F^{(1)}_{ijk} (P_i/r) Q'_j R'_k \Bigg|_{\substack{\theta = 0^\circ \\ \theta = 180^\circ}} = 0 \qquad \text{(E-3)}$$

Where

$$\left.\begin{array}{l} P'_1 = -P'_2 \qquad R'_1 = -R'_2 \\ P''_1 = -P''_2 \qquad R''_1 = -R''_2 \\ \\ Q_j\Big|_{\substack{\theta = 0^\circ \\ \theta = 180^\circ}} = \begin{cases} 1 & \text{odd i values} \\ 0 & \text{even j values} \end{cases} \\ \\ Q'_j\Big|_{\substack{\theta = 0^\circ \\ \theta = 180^\circ}} = \begin{cases} 1 & \text{even j values} \\ 0 & \text{odd j values} \end{cases} \end{array}\right\} \qquad \text{(E-4)}$$

Equations (E-1) through (E-4) give the following prescribed F^{I}_{ijk} values which are applied to each element at a crack location.

$$F^{(1)}_{ijk} = 0 \ , \qquad k \geq 3, \text{ all } i \text{ and } j$$

$$F^{(1)}_{ij1} = F^{(1)}_{ij2} \ , \qquad \text{all } i \text{ and } j$$

$$F^{(3)}_{ijk} = 0 \ , \qquad \text{all } j \text{ and } k, \ i \geq 3$$

$$F^{(3)}_{ijk} = 0 \ , \qquad i = 1,2, \quad j > 1$$

$$F^{(3)}_{ijk} = F^{(3)}_{2ik} \ , \qquad \text{all } k$$

APPENDIX F

INPUT DATA FOR THREE-DIMENSIONAL EQUILIBRIUM FINITE ELEMENT ANALYSIS

The analysis consists of four programs, PROA, PROB, PROC, and PROD. PROA read in the data to describe the problem and generates the integrals of the products of polynomials in the r and z directions. PROA also integrates the theta modes and stores them on tape. There is an option not to evaluate these integrals if they are already evaluated and stored on the tape. PROA also identifies the prescribed, independent and dependent coefficients of the stress functions.

PROB has no card input. Data is transferred to PROB from the tape. The purpose of PROB is to set up the matrix and right-side for the system of equations to be solved. PROC requires data to describe constraints or independent coefficients that are associated with the same element. These constraints arise from satisfying the stress-free boundary conditions at the free edge around the circular or elliptical hole. Constraints also arise in the case of the through-crack problem. The form of the constraints is given in Appendices B and C. The purposes of PROC is to solve the system of equations generated by PROB. This is done with extended core storage or with mass storage.

PROD evaluates stresses and strains at selected points in the laminate. These points are identified by input data. PROD also receives input data from the tape.

PROGRAM PROA

Variable	Format
JKEY, NCP	
JKEY = 1 for temperature problem.	
NCP [7 - through crack / 8 - no crack]. If temperature problem is required, read	
TEMPC	F10.6
TEMPC = Temperature change	
and ALP1(3), ALP2(3), ALP(3)	
ALP1(3), A2P2(3), A2P(3) are the coefficients of expansion	3(3F10.7/)
BETA	
BETA = 1 for circular hole problem. Otherwise a/b.	F12.9
OPT	I3
OPTQ = 1 for the Q's to be generated. If Q's need to be generated, read	
NTRE	I3
NTRE is the size of the Q matrix.	
RNX, RNY, RNXY. These are estimated stresses.	(3X,3F7.3)
IPRNT, ICHR, ICHZ	3I2

If (IPRNT .NE.0) program prints check information. This is used for checking operation of program and transfer of data. ICHR and ICHZ determines whether the integrals of R and Z should be recalculated. Both ICHR and ICHZ should have the same value. If a problem with the same element R and Z sizes that are on Tape 1 is being run, the integrals need not be recalculated and ICHR = ICHZ = 0. Else set both = 1.

RB(1), A(1)	2F10.0
RB(2), A(2)	2F10.0

RB(1), A(1) = Inner radius and R length of inner elements respectively. RB(2), A(2) = Inner radius and R length of outer elements respectively. RB(2) = RB(1) + A(1)

MMP(1), MMP(2)	2I2

MMP(1) and MMP(2) are minus the starting powers of R in the series for the inner and outer elements respectively. Generally MMP(1) = 2 and MMP(2) = 2 or 0 are used for circular holes.

Variable	Format
LNR (set equal to zero)	I2
NTZ (set equal to four)	I2
MMR(1), MMR(2), MMR(3)	3I2

MMR(1), MMR(2) and MMR(3) are minus the starting powers of the z-series terms going from the center to the outer plies respectively.

LNRZ (set equal to zero)	I2
ZB(1), H(1)	2F10.0
ZB(2), H(2)	2F10.0
ZB(3), H(3)	2F10.0

ZB and H are the starting local coordinates for the z-axis on the element and the thickness of each ply (starting with the center ply(1) and going to the outer ply(3)) respectively. If MMR(I) .GT.0, then ZB(I) .GT.0.

The following data describes the theta terms in the series. In the following description, the relation between N and the theta terms is

$$N \geq 1 \text{ and odd} \qquad N \geq 2 \text{ and even}$$

$$Q(\theta) = (\cos\theta)^{N-1} \qquad Q(\theta) = (\cos\theta)^{N-1}\sin\theta$$

These data are read in for each stress function, and for the theta variation at (1) the hole, (2) between the inner and outer elements, and (3) at the outer edge of the outer element.

The order is

Stress Function No. 1

Location A. Around the hole $\left\{\begin{array}{l}\phi \text{ variation}\\ \phi_r \text{ variation}\\ \phi_{rr} \text{ variation}\end{array}\right.$

Location B. Between inner and outer elements $\left\{\begin{array}{l}\phi \text{ variation}\\ \phi_r \text{ variation}\\ \phi_{rr} \text{ variation}\end{array}\right.$

Location C. Outer edge of outer element $\left\{\begin{array}{l}\phi \text{ variation}\\ \phi_r \text{ variation}\\ \phi_{rr} \text{ variation}\end{array}\right.$

Stress Function No. 2

Same order as Stress Function No. 1

Stress Function No. 3

Same order as above

Variable	Format

For each stress function and each location and each ϕ, ϕ_r and ϕ_{rr}, two cards are required. These are

NTV	I2

NTV is the number of terms in the theta variation series.

NTV $\leq$ 18 and (IDT (ISF,I,J), J = 1, NTV).	24I2

IDT1 (ISF,I,J) is the value of N described in the preceding. ISF = 1, 2, or 3. I goes from 1 to 6 and denotes ϕ, ϕ_r, or ϕ_{rr} variations.

These two data card formats will be repeated a total of 3ISF-s x 3 locations x 3 (variations on ϕ, ϕ_r, or ϕ_{rr}) = 27 times.

The next set of data describes the strain-stress relations and the ply singles. For each ply, starting with the center and working to the outer ply, the following data are needed.

PHI	F10.0

PHI is the ply angle. with respect to the θ equals zero line.

PROGRAM PROB has no input data.

PROGRAM PROC

The data for PROC is to put constraints on the stress function coefficients that could not easily be handled elsewise. These constraints are of the form, $X_i = Y_i \pm Z_i$, or $X_i = -Y_i$. Other constraints of the form $A_i = B_i$ are handled automatically within the program. The data read in are (1) the number of equations of the form $X_i = Y_i \pm Z_i$ or $X_i = -Y_i$, and (2) identifiers for X_i, $\pm Y_i$, and $\pm Z_i$. The input is as follows.

NEQ	I2

NEQ is the number of equations of the form $X_i = Y_i \pm Z_i$ or $X_i = -Y_i$ to be satisfied.

(J,DEP(K), IN1(K), IN2(K), K = 1, NEQ)	I2,8X,3(I6,4X)

J is not used by program. It is a means to number the input data cards. DEP, IN1, and IN2 have the value K + 10J + 100I + 1000ISF + 10000 EL where the element number is EL, the stress function number is ISF and the coefficient is F_{IJK} of the term $F_{IJK}\ P_I(R)Q_J(\theta)R_K(Z)$.

A minus sign in front of IN1 or IN2 indicates there is a negative sign in the equation. DEP is always positive. IN1 may be positive or negative but not zero. IN2 may be positive, negative or zero.

Variable	Format

PROGRAM PROD

IREAD, IPUNCH	2I2

IREAD = 1 if solution vector is supplied on cards, else IREAD = 0. IPUNCH = 1 if solution is to be punched out on cards, else IPUNCH = 0.

If IREAD = 1, then supply

NC	I10

NC = Number of coefficients in solution

(SV(I), I = 1, NC)	(2X,4E16.8)

SV(I) = solutions vector.

Comment: The following is data describing locations in the laminate that stresses are to be printed out.

L	I2

L = Element No. $1 \leq L \leq 6$. If L = 0, program stops.

NRBAR	I2

NRBAR = Number of radii to calculate stresses at $1 \leq$ NRBAR ≤ 11.

(RBAR(I), I = 1, NRBAR)	11F5.0
NTBAR	I2

NTBAR = Number of thetas to calculate stresses at $1 \leq$ NTBAR ≤ 21.

(TBAR(I), I = 1, NTBAR)	11F5.0

Values of theta, no more than 11 per card.

NZBAR	I2

NZBAR = Number of Z locations to calculate stresses at $1 \leq$ NZBAR ≤ 11.

(ZBAR(I), I = 1, NZBAR)	11F5.0

Z locations.

All six stresses are calculated at the each of the R, θ, and Z points described by input. To compute stresses for another set of points, repeat input, starting with L. Last card in data for PROD should be a blank card.

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